

1. The curve C has the equation $2x + 3y^2 + 3x^2y = 4x^2$.
 The point P on the curve has coordinates $(-1, 1)$.

(a) Find the gradient of the curve at P .

(5)

(b) Hence find the equation of the normal to C at P , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers.

(3)

$$a) \frac{d}{dx}(2x) + \frac{d}{dx}(3y^2) + \frac{d}{dx}(3x^2y) = \frac{d}{dx}(4x^2)$$

$$\Rightarrow 2 + 6y \frac{dy}{dx} + 6xy + 3x^2 \frac{dy}{dx} = 8x$$

$$\Rightarrow (6y + 3x^2) \frac{dy}{dx} = 8x - 6xy - 2$$

$$\therefore \frac{dy}{dx} = \frac{8x - 6xy - 2}{6y + 3x^2}$$

$$\text{at } (-1, 1) \quad M_t = \frac{-8 + 6 - 2}{6 + 3} = \frac{-4}{9}$$

$$b) M_n = \frac{9}{4} \Rightarrow y - 1 = \frac{9}{4}(x + 1)$$

$$(-1, 1) \quad \therefore 9x - 4y + 13 = 0$$

2. (a) Use integration by parts to find $\int x \sin 3x \, dx$.

(b) Using your answer to part (a), find $\int x^2 \cos 3x \, dx$.

$$\begin{aligned} \int x \sin 3x \, dx &= -\frac{1}{3}x \cos 3x - \int -\frac{1}{3} \cos 3x \, dx & u &= x & v &= -\frac{1}{3} \cos 3x \\ & & u' &= 1 & v' &= \sin 3x \\ &= -\frac{1}{3}x \cos 3x + \frac{1}{9} \sin 3x + C \end{aligned}$$

$$\begin{aligned} \text{b) } \int x^2 \cos 3x \, dx &= \frac{1}{3}x^2 \sin 3x - \frac{2}{3} \int x \sin 3x \, dx & u &= x^2 & v &= \frac{1}{3} \sin 3x \\ & & u' &= 2x & v' &= \cos 3x \end{aligned}$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3} \left[-\frac{1}{3} \cos 3x + \frac{1}{9} \sin 3x \right] + C$$

$$= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x \cos 3x - \frac{2}{27} \sin 3x + C$$

3. (a) Expand

$$\frac{1}{(2-5x)^2}, \quad |x| < \frac{2}{5}$$

in ascending powers of x , up to and including the term in x^2 , giving each term as a simplified fraction.

Given that the binomial expansion of $\frac{2+kx}{(2-5x)^2}$, $|x| < \frac{2}{5}$, is

$$\frac{1}{2} + \frac{7}{4}x + Ax^2 + \dots$$

(b) find the value of the constant k ,

(2)

(c) find the value of the constant A .

(2)

$$a) (2-5x)^{-2} = 2^{-2} \left(1 - \frac{5}{2}x\right)^{-2} = \frac{1}{4} \left(1 - \frac{5}{2}x\right)^{-2}$$

$$= \frac{1}{4} \left(1 - 2\left(-\frac{5}{2}x\right) + \frac{(-2)(-3)}{2} \left(-\frac{5}{2}x\right)^2 \right)$$

$$= \frac{1}{4} + \frac{5}{4}x + \frac{75}{16}x^2$$

$$b) \frac{2+kx}{(2-5x)^2} = (2+kx) \left(\frac{1}{4} + \frac{5}{4}x + \frac{75}{16}x^2 + \dots \right)$$

$$\begin{array}{r} \textcircled{\times 2} \\ \textcircled{\times kx} \end{array} \quad \begin{array}{r} \frac{2}{4} + \frac{10}{4}x + \frac{150}{16}x^2 \\ \frac{1}{4}kx + \frac{5}{4}kx^2 + \dots \end{array}$$

$$= \frac{1}{2} + \left(\frac{5}{2} + \frac{1}{4}k\right)x + \left(\frac{75}{8} + \frac{5}{4}k\right)x^2$$

$$\therefore \frac{5}{2} + \frac{1}{4}k = \frac{7}{4} \Rightarrow \frac{1}{4}k = -\frac{3}{4} \Rightarrow \underline{k = -3}$$

$$\therefore A = \frac{75}{8} + \frac{5}{4}(-3) = \frac{45}{8}$$

4.

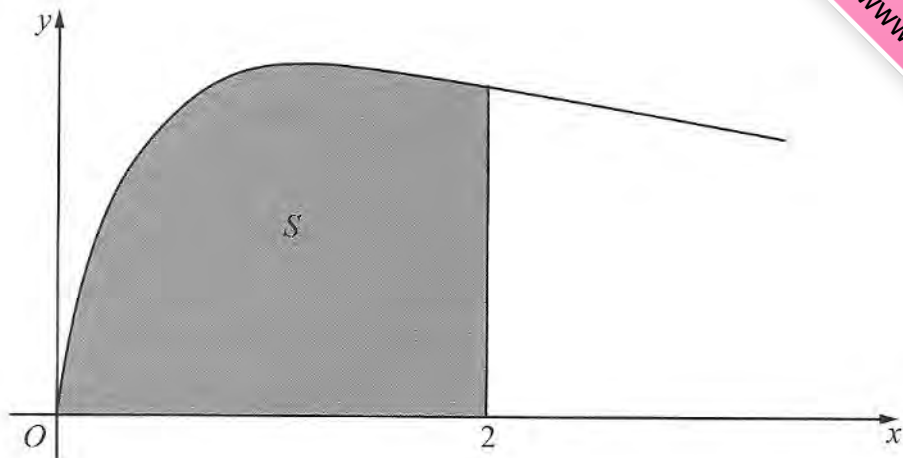


Figure 1

Figure 1 shows the curve with equation

$$y = \sqrt{\left(\frac{2x}{3x^2 + 4}\right)}, \quad x \geq 0$$

The finite region S , shown shaded in Figure 1, is bounded by the curve, the x -axis and the line $x = 2$

The region S is rotated 360° about the x -axis.

Use integration to find the exact value of the volume of the solid generated, giving your answer in the form $k \ln a$, where k and a are constants.

(5)

$$\text{Volume} = \pi \int_0^2 y^2 dx = \pi \int_0^2 \frac{2x}{3x^2+4} dx = \frac{1}{3} \pi \int_0^2 \frac{6x}{3x^2+4} dx$$

$$= \frac{1}{3} \pi \left[\ln(3x^2+4) \right]_0^2 = \frac{1}{3} \pi (\ln 16 - \ln 4)$$

$$= \frac{1}{3} \pi (\ln 4) = \frac{1}{3} \pi (2 \ln 2) = \frac{2}{3} \pi \ln 2$$

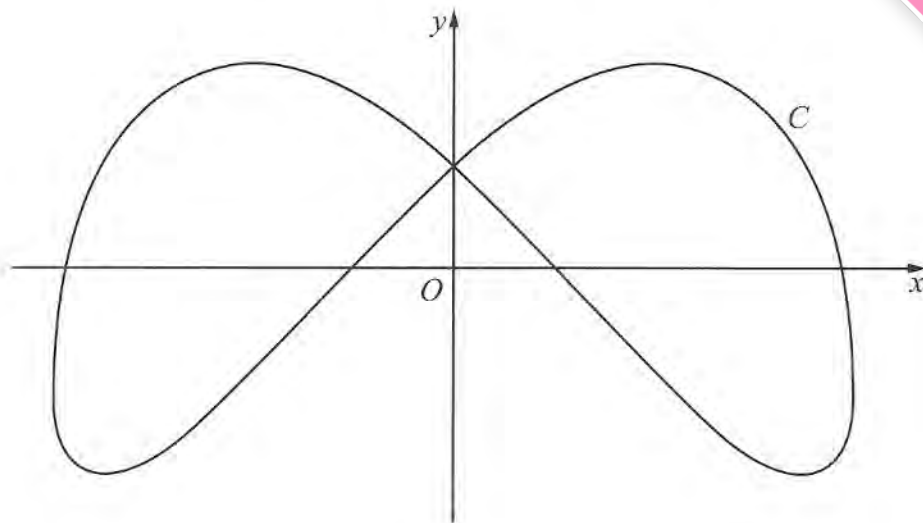


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 4 \sin\left(t + \frac{\pi}{6}\right), \quad y = 3 \cos 2t, \quad 0 \leq t < 2\pi$$

$$0 \leq 2t < 4\pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t .

(3)

(b) Find the coordinates of all the points on C where $\frac{dy}{dx} = 0$

(5)

$$\frac{dx}{dt} = 4 \cos\left(t + \frac{\pi}{6}\right) \quad \frac{dy}{dt} = -6 \sin 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-6 \sin 2t}{4 \cos\left(t + \frac{\pi}{6}\right)}$$

$$b) \frac{dy}{dx} = 0 \Rightarrow -6 \sin 2t = 0 \Rightarrow \sin 2t = 0$$

$$2t = 0, \pi, 2\pi, 3\pi, \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$t = 0 \quad (2, 3) \quad t = \frac{\pi}{2} \quad (2\sqrt{3}, -3) \quad t = \pi \quad (-2, 3) \quad t = \frac{3\pi}{2} \quad (-2\sqrt{3}, -3)$$

6.

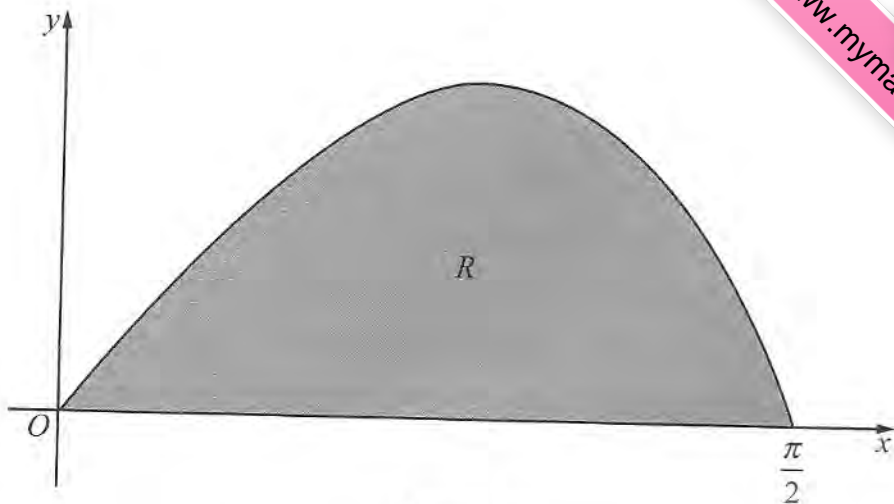


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \frac{2 \sin 2x}{1 + \cos x}$, $0 \leq x \leq \frac{\pi}{2}$.

The finite region R , shown shaded in Figure 3, is bounded by the curve and the x -axis.

The table below shows corresponding values of x and y for $y = \frac{2 \sin 2x}{1 + \cos x}$.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.73508	1.17157	1.02280	0

- (a) Complete the table above giving the missing value of y to 5 decimal places. (1)
- (b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R , giving your answer to 4 decimal places. (3)
- (c) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2 \sin 2x}{1 + \cos x} dx = 4 \ln(1 + \cos x) - 4 \cos x + k$$

where k is a constant. (5)

- (d) Hence calculate the error of the estimate in part (b), giving your answer to 2 significant figures. (3)

www.mymathscloud.com
My Maths Cloud

$$\text{Area} = \frac{1}{2} \left(\frac{\pi}{8} \right) [0+0+2(0.73508+1.17157+1.0)]$$

$$\approx 1.1504 \text{ (4dp)}$$

c) $u = 1 + \cos 2x \quad \frac{du}{dx} = -\sin 2x \Rightarrow dx = -\frac{du}{\sin 2x}$

$$2 \sin 2x = 4 \sin x \cos x \quad \cos x = u - 1$$

$$\therefore \int \frac{2 \sin 2x}{1 + \cos 2x} dx = 4 \int \frac{\sin x (u-1)}{u} - \frac{du}{\sin 2x}$$

$$= -4 \int \frac{u-1}{u} du = -4 \int 1 - \frac{1}{u} du$$

$$= -4 [u - \ln u + c]$$

$$= 4 \ln u - 4u + u = 4 \ln(1 + \cos x) - 4(1 + \cos x) + u$$

d) $4 \left[\ln(1 + \cos x) - (1 + \cos x) \right] \Big|_0^{\frac{\pi}{2}}$

$$= 4 \left[(\ln(1+0) - (1+0)) - (\ln(1+1) - (1+1)) \right]$$

$$= 4 [1 - \ln 2] \approx 1.227411278 \dots$$

$$\therefore \text{error} = -0.077 \text{ (2sf)}$$

(Note - sign probably doesn't matter)

7. Relative to a fixed origin O , the point A has position vector $(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$, the point B has position vector $(5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k})$, and the point D has position vector $(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$.

The line l passes through the points A and B .

- (a) Find the vector \vec{AB} . (2)
- (b) Find a vector equation for the line l . (2)
- (c) Show that the size of the angle BAD is 109° , to the nearest degree. (4)

The points A , B and D , together with a point C , are the vertices of the parallelogram $ABCD$, where $\vec{AB} = \vec{DC}$.

- (d) Find the position vector of C . (2)
- (e) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures. (3)
- (f) Find the shortest distance from the point D to the line l , giving your answer to 3 significant figures. (2)

$$a = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} \quad d = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{AB} = b - a = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \quad l = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$$

$$c) \hat{BAD} = \cos^{-1} \left(\frac{\vec{AB} \cdot \vec{AD}}{|\vec{AB}| |\vec{AD}|} \right) \quad \vec{AD} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

$$\hat{BAD} = \cos^{-1} \left(\frac{-8}{\sqrt{43} \sqrt{14}} \right)$$

$$= 109.0295\dots$$

$$\approx 109^\circ \text{ (nd)}$$

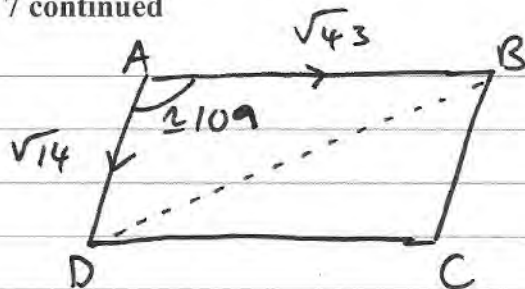
$$|\vec{AB}| = \sqrt{3^2 + 3^2 + 5^2} = \sqrt{43}$$

$$|\vec{AD}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

$$\vec{AB} \cdot \vec{AD} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

$$= -9 + 6 - 5 = -8$$

Question 7 continued



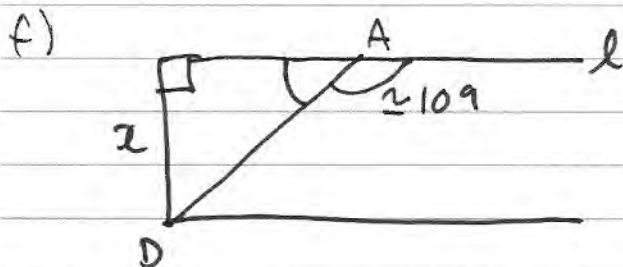
$$c = d + \overline{AB}$$

$$c = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}$$

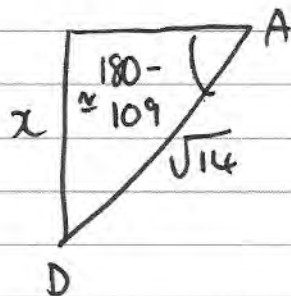
e) Area ABCD = 2 × Area $\triangle BAD$

$$= 2 \times \frac{1}{2} (\sqrt{43})(\sqrt{14}) \sin(109.029\dots)$$

$$\approx 23.2 \text{ (3sf)}$$



x = shortest distance
l to D.



$$x = \sqrt{14} \sin(180 - 109.029\dots)$$

$$\approx 3.54 \text{ (3sf)}$$

8. (a) Express $\frac{1}{P(5-P)}$ in partial fractions.

A team of conservationists is studying the population of meerkats on a natural reserve. The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{15}P(5-P), \quad t \geq 0$$

where P , in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + ce^{-\frac{1}{3}t}}$$

where a , b and c are integers.

(8)

(c) Hence show that the population cannot exceed 5000

(1)

$$\begin{aligned} \text{a) } \frac{1}{P(5-P)} &= \frac{A}{P} + \frac{B}{5-P} \Rightarrow 1 = A(5-P) + B(P) \\ P=5 &\Rightarrow 1 = 5B \quad B = \frac{1}{5} \\ P=0 &\Rightarrow 1 = 5A \quad A = \frac{1}{5} \end{aligned}$$

$$\therefore \frac{1}{5P} + \frac{1}{5(5-P)}$$

$$\text{b) } \int \frac{1}{P(5-P)} dP = \int \frac{1}{15} dt \Rightarrow \frac{1}{5} \int \frac{1}{P} + \frac{1}{5-P} dP = \frac{1}{15} \int 1 dt$$

$$\Rightarrow \int \frac{1}{P} + \frac{1}{5-P} dP = \frac{1}{3} \int 1 dt \Rightarrow \ln P - \ln(5-P) = \frac{1}{3}t + C$$

$$\Rightarrow \ln\left(\frac{P}{5-P}\right) = \frac{1}{3}t + C$$

$$t=0, P=1 \Rightarrow \ln\left(\frac{1}{4}\right) = C \Rightarrow C = -\ln 4$$

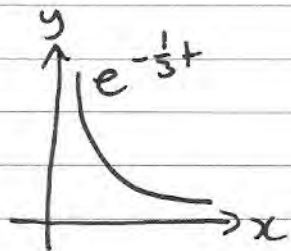
$$\Rightarrow \ln\left(\frac{P}{5-P}\right) = \frac{1}{3}t - \ln 4$$

$$\frac{P}{S-P} = e^{\frac{1}{3}t - \ln 4} = e^{\frac{1}{3}t} \div e^{\ln 4} = \frac{e^{\frac{1}{3}t}}{4}$$

$$\Rightarrow 4P = 5e^{\frac{1}{3}t} - Pe^{\frac{1}{3}t} \Rightarrow P(4 + e^{\frac{1}{3}t}) = 5e^{\frac{1}{3}t}$$

$$\Rightarrow P = \frac{5e^{\frac{1}{3}t}}{4 + e^{\frac{1}{3}t}} \quad \begin{array}{l} \div e^{\frac{1}{3}t} \\ \div e^{\frac{1}{3}t} \end{array} \Rightarrow P = \frac{5}{4e^{-\frac{1}{3}t} + 1}$$

$$\underline{a=5} \quad \underline{b=1} \quad \underline{c=4}$$



as $x \rightarrow \infty$ $e^{-\frac{1}{3}t} \rightarrow 0$

$$\Rightarrow 4e^{-\frac{1}{3}t} + 1 \rightarrow 1$$

$$4e^{-\frac{1}{3}t} + 1 > 1$$

$$\therefore P \rightarrow \frac{5}{1} \rightarrow 5$$

$$P < 5$$

P is in thousands \therefore Population can not exceed 5000